A note on the modulus of continuity of a periodic function

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\textbf{A B S T R A C T}

Let \( f(x) \) be a periodic function with period \( T \). In Rivlin (1969) \cite{1} it is claimed that the modulus of continuity is independent of \( a \) on \([a, a + T]\). In this note we show that this is not correct.

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1. Introduction

\textbf{Definition 1.1} \cite{1}. Let \( f(x) \) be defined on \([a, b]\); the modulus of a function on \([a, b]\), \( w(\delta) \), is defined for \( \delta > 0 \) by

\[
w(\delta) = \sup_{x_1, x_2 \in [a, b]} |f(x_1) - f(x_2)|.
\]

\( w(\delta) \) is shorthand for \( w(f; [a, b]; \delta) \).

It is said to have certain properties (see \cite{1}); for example:

1. If \( 0 < \delta_1 \leq \delta_2 \), then \( w(\delta_1) \leq w(\delta_2) \).
2. \( f(x) \) is uniformly continuous on \([a, b]\) if and only if

\[
\lim_{\delta \to 0} w(\delta) = 0.
\]

3. If \( \lambda > 0 \), then

\[
w(\lambda \delta) \leq (1 + \lambda)w(\delta).
\]

Also:

4. If \( f(x) \) has period \( T \), \( w(f; [a, a + T]; \delta) \) is independent of \( a \).

Now, using a counterexample we show that the last property is not correct.

\textbf{Example 1.} Let

\[
f(x) = \begin{cases} 
-x + 1 & 0 \leq x < 0.9 \\
x - 8 & 0.9 \leq x \leq 1
\end{cases}
\]

and \( f(x + 1) = f(x) \). It is clear that \( f \) is periodic with period 1.
Let $\delta = 0.1$; then

$$w(f; [0, 1]; 0.1) = 0.9$$

and

$$w(f; [0.95; 1.95]; 0.1) = 0.45.$$  

References